0	1	2	3	4	5	6	7	8
10	10	10	10	10	10	8	10	10

8,8

1a. Formulate the principle of mathematical induction.

Let S_n be a statement about the positive integer n. Suppose that:

1) S_1 is true.
2) S_{n+1} is true whenever S_n is true ($\forall n > 0$).

Then S_n is true for all positive integers n.

b. Use mathematical induction to show that $r^0 + r^1 + \dots r^{n-1} = \frac{r^n - 1}{r-1}$ for every natural number $n \geq 1$ and $r \neq 1$.

Statement: $r^0 + r^1 + r^2 + \dots + r^{n-1} = \frac{r^n - 1}{r-1}$ $\forall n \geq 1$ and $r \neq 1$

Proof by mathematical induction:

1) Base step: fill in $n = 1$

$$S_1 = r^{n-1} = r^{1-1} = r^0 = 1 \quad \&$$
$$S_1 = \frac{r^{\cancel{n}}-1}{r-1} = \frac{r^{\cancel{1}}-1}{r-1} = \frac{r-1}{r-1} = 1$$

2) Induction step: assume that S_n holds, prove that S_{n+1} holds.

$$S_n: r^0 + r^1 + r^2 + \dots + r^{n-1} = \frac{r^n - 1}{r-1}$$
$$S_{n+1}: \underbrace{r^0 + r^1 + r^2 + \dots + r^{n-1}} + r^{(n+1)-1} = \frac{r^{(n+1)} - 1}{r-1}$$
$$\frac{r^n-1}{r-1} + r^n = \frac{r^{(n+1)}-1}{r-1}$$
$$\frac{r^n-1}{r-1} + \frac{r^n(r-1)}{r-1} = \frac{r^{(n+1)}-1}{r-1}$$
$$\frac{r^n - 1 + r^{n+1} - r^n}{r-1} = \frac{r^{(n+1)}-1}{r-1}$$
$$\frac{-1 + r^{n+1}}{r-1} = \frac{r^{n+1}-1}{r-1}$$
$$\frac{r^{n+1}-1}{r-1} = \frac{r^{n+1}-1}{r-1} \quad \text{if } n \geq 1 \text{ and } r \neq 1$$

Thus, by mathematical induction, we have proven that the statement $S_n: r^0 + r^1 + r^2 + \dots + r^{n-1} = \frac{r^n-1}{r-1}$ holds.

Q.E.D.

2. Find all (complex) solutions z of $(z+i)^3 = \frac{\sqrt{2}}{1-i}$ and plot them in the complex plane.

First, we have to find the $re^{i\theta}$-form of $\frac{\sqrt{2}}{1-i}$:

$$\frac{\sqrt{2}}{1-i} = \frac{\sqrt{2}}{1-i}\cdot\frac{1+i}{1+i} = \frac{\sqrt{2}(1+i)}{(1-i)(1+i)} = \frac{\sqrt{2}(1+i)}{1-i^2} = \frac{\sqrt{2}(1+i)}{2} = \tfrac{1}{2}\sqrt{2}(1+i) = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\,i$$

$$r = \sqrt{a^2+b^2} = \sqrt{(\tfrac{1}{2}\sqrt{2})^2 + (\tfrac{1}{2}\sqrt{2})^2} = \sqrt{(\tfrac{1}{4}\cdot 2) + (\tfrac{1}{4}\cdot 2)} = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}} = \sqrt{1} = 1$$

$$\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}}\right) = \arctan(1) = \tfrac{1}{4}\pi + 2k\pi \quad \text{where } k = 0,1,2,3\ldots$$

So: $\frac{\sqrt{2}}{1-i} = 1\cdot e^{i(\frac{1}{4}\pi + 2k\pi)}$

$$\Rightarrow \quad (z+i)^3 = 1\cdot e^{i(\frac{1}{4}\pi + 2k\pi)}$$

$$z + i = \sqrt[3]{1}\cdot e^{i(\frac{1}{12}\pi + \frac{2}{3}k\pi)}$$

$$z = e^{i(\frac{1}{12}\pi + \frac{2}{3}k\pi)} - i \qquad \text{where } k = 0,1,2,3,\ldots$$

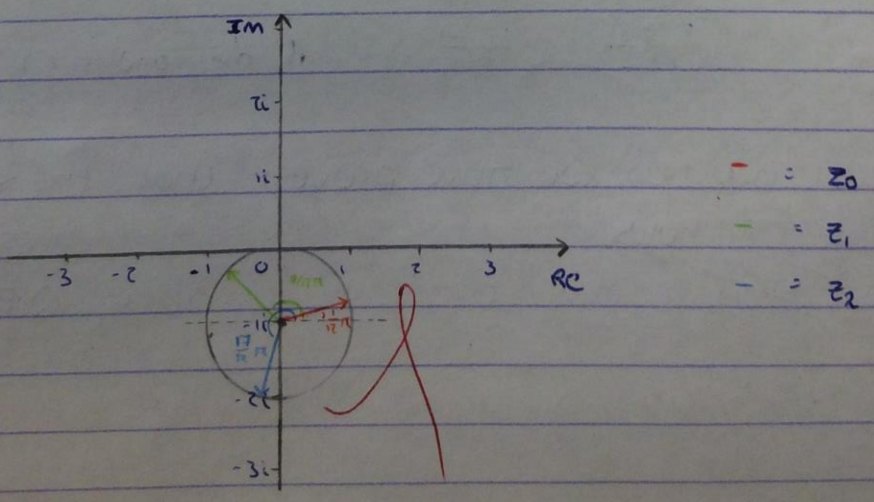
This yields the following solutions:

$$z_0 = e^{i(\frac{1}{12}\pi)} - i$$

$$z_1 = e^{i(\frac{9}{12}\pi)} - i$$

$$z_2 = e^{i(\frac{17}{12}\pi)} - i$$

Sketching these solutions then gives:



3a. The function f is defined on some open interval that contains a number a, except possibly at a itself. Give the precise definition of $\lim_{x \to a} f(x) = L$.

$\lim_{x \to a} f(x) = L$ $\quad$ $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

b. Prove, using this definition, that $\lim_{x \to 4} (2x - 5) = 3$

$\lim_{x \to 4} (2x - 5) = 3$ $\quad$ $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-4| < \delta \Rightarrow |2x - 5 - 3| < \varepsilon$

$|2x - 5 - 3| < \varepsilon \quad \wedge \quad 0 < |x - 4| < \delta$

$|2x - 8| < \varepsilon$

$|2(x - 4)| < \varepsilon$

$2|x - 4| < \varepsilon$

$|x - 4| < \frac{\varepsilon}{2}$

So, we found: $\frac{\varepsilon}{2} = \delta$. Filling this in for δ yields:

$0 < |x - 4| < \delta$

$0 < |x - 4| < \frac{\varepsilon}{2}$

$0 < 2|x - 4| < \varepsilon$

$0 < |2x - 8| < \varepsilon$

$0 < |2x - 5 - 3| < \varepsilon$ $\quad$?

We found that $\frac{\varepsilon}{2} = \delta$.

Thus, by using the precise definition of a limit, we found a δ correspon to an ε, so our conditions are satisfied. Thus: $\lim_{x \to 4} (2x - 5) = 3$.

Q.E.D.

4a. Formulate the Mean Value Theorem

Let f be a function defined on the interval $[a,b]$ that satisfies the following hypotheses:

1) f is continuous on the closed interval $[a,b]$.
2) f is differentiable on the open interval (a,b).

Then there exists a number c in (a,b) such that:

4

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \text{where } c \in (a,b)$$

b. Let f be a function that is differentiable everywhere and $f(-x) = -f(x)$ for all x. Show that for every number b, there exists a number c in $(-b,b)$ such that $f'(c) = \frac{f(b)}{b}$

We are given that: $f(-x) = -f(x)$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$

Suppose we take an interval of ~~(b)~~ $(-b,b)$. According to the Mean Value Theorem, then there exists a c in $(-b,b)$ such that:

$$f'(c) = \frac{f(b)-f(-b)}{b-(-b)} \qquad \text{where } c \in (-b,b)$$

We are given that $f(-x) = -f(x)$, so: $f(-b) = -f(b)$. This then yields:

$$f'(c) = \frac{f(b)-(-f(b))}{b+b}$$
$$= \frac{f(b)+f(b)}{2b}$$
$$= \frac{2f(b)}{2b}$$
$$= \frac{f(b)}{b}$$

Thus, by using the Mean Value Theorem, we have proven that for $f(-x) = -f(x)$ there exists a number c in $(-b,b)$ such that: $f'(c) = \frac{f(b)}{b}$

6

Q.E.D.

5a. If f is a differentiable function and $g(x) = xf(x)$, use the definition of derivative to show that $g'(x) = xf'(x) + f(x)$

Definition of derivative: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Using this definition yields for $g(x) = xf(x)$:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)f(x+h) - xf(x)}{h}$$

5

$$= \lim_{h\to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}$$

$$= \lim_{h\to 0} \frac{xf(x+h) - xf(x)}{h} + \lim_{h\to 0} \frac{hf(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{x(f(x+h)-f(x))}{h} + \lim_{h\to 0} f(x+h)$$

$$= x \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + f(x)$$

$$= xf'(x) + f(x)$$

Thus, by using the definition of derivative, we have proven that $g'(x) = xf'(x) + f(x)$ (for $g(x) = xf(x)$). Q.E.D.

b. Use the definition of derivative to show that the derivative of the exponential function $f(x) = a^x$ satisfies $f'(x) = f'(0)a^x$ (where $a>0$)

Definition of derivative: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Using this definition yields for $f(x) = a^x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{(x+h)} - a^x}{h}$$

5

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x(a^h-1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{(a^h-1)}{h}$$

$$= a^x \lim_{h\to 0} \left(\frac{f(h)-f(0)}{h}\right) = a^x \cdot f'(0) \quad \text{where } a>0$$

Thus, by using the definition of derivative, we have proven that $f'(x) = f'(0)a^x$ for $f(x) = a^x$

Q.E.D.

6a. Evaluate $\lim_{x \to 0^+} x^{\sqrt{x}}$

3

In order to evaluate this limit, we have to rewrite it:

$$\lim_{x \to 0^+} x^{\sqrt{x}} = \lim_{x \to 0^+} (e^{\ln x})^{\sqrt{x}} = \lim_{x \to 0^+} (e^{\sqrt{x}\ln x})$$

This limit can be solved: show

$$\lim_{x \to 0^+} (e^{\sqrt{x}\ln x}) = e^{\lim_{x \to 0^+} (\sqrt{x}\ln x)} = e^0 = 1$$

b. Evaluate $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{-t^2} dx$

5

To work this limit out, we have to consider $\int_0^x e^{-t^2} dx$ first. From logical thinking, we can state that when x tends to zero, the integral of a function $f(x)$ from 0 to x (so $\int_0^x f(x')dx'$) also tends to zero. So, $\lim_{x \to 0} \int_0^x e^{-t^2} dx = 0$.

Combining this with $\lim_{x \to 0} \frac{1}{x}$, we get:

$$\lim_{x \to 0} \frac{1}{x} \int_0^x e^{-t^2} dx = \text{“}\frac{0}{0}\text{”}$$

This indetermined form can be solved by using Bernoulli's Rule; in combination with the Fundamental Theorem of Calculus part 1, which states that: if $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$. Thus, we get:

$$\lim_{x \to 0} \frac{1}{x} \int_0^x e^{-t^2} dx = \lim_{x \to 0} \frac{\int_0^x e^{-t^2} dx}{x}$$

$$\overset{BR}{=} \lim_{x \to 0} \frac{e^{-x^2}}{1}$$

$$= \lim_{x \to 0} e^{-x^2}$$

$$= e^{\lim_{x \to 0}(-x^2)}$$

$$= e^0 = 1$$

Thus, by using Bernoulli's Rule (or L'Hopital's Rule) and the Fundamental Theorem of Calculus part 1, we found that:

$$\lim_{x \to 0} \frac{1}{x} \int_0^x e^{-t^2} dt = 1$$

7a. Evaluate $\int e^x \sqrt{1+e^x}\,dx$

In this case, we can use substitution:

$u = 1 + e^x \rightarrow \frac{du}{dx} = e^x \rightarrow dx = \frac{du}{e^x}$

Using this then yields:

$$\begin{aligned}\int e^x \sqrt{1+e^x}\,dx &= \int e^x \sqrt{u}\,\frac{du}{e^x} \\ &= \int \sqrt{u}\,du \\ &= \tfrac{2}{3}u\sqrt{u} + C \qquad \text{where } C \in \mathbb{R} \\ &= \tfrac{2}{3}(1+e^x)\sqrt{1+e^x} + C\end{aligned}$$

5

Thus, by using substitution, we found: $\int e^x \sqrt{1+e^x}\,dx = \frac{2}{3}(1+e^x)\sqrt{1+e^x} + C$

b. Evaluate $\int e^x \sin x\,dx$

In this case, we can use integration by parts. This technique uses the following general formula:

$\int fg'\,dx = fg - \int f'g\,dx$

5

Applying this on our function then yields:

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$ (where $g' = e^x$ and $f = \sin x$)

Applying integration by parts again yields:

$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$ (where $g' = e^x$ and $f = \cos x$)

Combining these results then gives:

$$\begin{aligned}\int e^x \sin x\,dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx \\ 2\int e^x \sin x\,dx &= e^x \sin x - e^x \cos x + C \qquad \text{where } C \in \mathbb{R} \\ \int e^x \sin x\,dx &= \tfrac{1}{2}(e^x \sin x - e^x \cos x) + C = \tfrac{1}{2}e^x(\sin x - \cos x) + C\end{aligned}$$

Thus, by using integration by parts, we found: $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$

8. Solve the differential equation $y' + 3x^2y = 6x^2$

In order to solve this differential equation, we can make use of the integration factor. For a differential equation in the form of $y' + P(x)y = Q(x)$, an integration factor works as follows:

1) Find the integration factor I by using $(Iy)' = I(y' + P(x)y)$
2) Solve, using this factor, $(Iy)' = IQ(x)$ for y.

We can apply this technique to $y' + 3x^2y = 6x^2$:

1) Find the integration factor I by using $(Iy)' = I(y' + P(x)y)$

$$(Iy)' = I(y' + P(x)y)$$
$$I'y + Iy' = Iy' + P(x)yI$$
$$I'\cancel{y} = P(x)\cancel{y}I$$
$$\frac{dI}{dx} = P(x)I \qquad \text{where } P(x) = 3x^2$$
$$\int \frac{1}{I}\,dI = \int 3x^2\,dx$$
$$\ln|I| = x^3 + C_1 \qquad \text{where } C_1 \in \mathbb{R}$$
$$|I| = e^{x^3 + C_1}$$
$$|I| = e^{C_1} \cdot e^{x^3}$$
$$I = C_2 \cdot e^{x^3} \qquad \text{where } C_2 \in \mathbb{R}$$

So: $I = e^{x^3}$ (if $C_2 = 1$)

2) Solve, using this factor, $(Iy)' = IQ(x)$ for y.

$$(Iy)' = IQ(x)$$
$$(Iy)' = e^{x^3}(6x^2)$$
$$Iy = \int e^{x^3} 6x^2\,dx$$
$$e^{x^3}y = 2e^{x^3} + C_3 \qquad \text{where } C_3 \in \mathbb{R}$$
$$y = \frac{2e^{x^3} + C_3}{e^{x^3}}$$
$$= 2 + \frac{C_3}{e^{x^3}}$$
$$= 2 + \frac{C}{e^{x^3}} \qquad \text{where } C_3 \equiv C \quad (C \in \mathbb{R})$$

Thus, by using the method of integration factor, we found:

$y = 2 + \frac{C}{e^{x^3}}$. $(C \in \mathbb{R})$